41^{st} International Mathematical Olympiad

Taejon, Republic of Korea

Day I 9 a.m. - 1:30 p.m.

July 19, 2000

1. Two circles ω_1 and ω_2 intersect at M and N. Line ℓ is tangent to the circles at A and B, respectively, so that M lies closer to ℓ than N. Line CD, with C on ω_1 and D on ω_2, is parallel to ℓ and passes through M. Let lines AC and BD meet at E; let lines AN and CD meet at P; and let lines BN and CD meet at Q. Prove that $EP = EQ$.

2. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$(a - 1 + 1/b)(b - 1 + 1/c)(c - 1 + 1/a) \leq 1.$$

3. Let $n \geq 2$ be a positive integer. Initially, there are n fleas on a horizontal line, not all at the same point. For a positive real number λ, define a *move* as follows:

   choose any two fleas, at points A and B, with A to the left of B;
   let the flea at A jump to the point C on the line to the right of B
   with $BC/AB = \lambda$.

   Determine all values of λ such that, for any point M on the line and any initial positions of the n fleas, there is a finite sequence of moves that will take all the fleas to positions to the right of M.

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4. A magician has one hundred cards numbered 1 to 100. He puts them into three boxes, a red one, a white one and a blue one, so that each box contains at least one card. A member of the audience selects two of the three boxes, chooses one card from each and announces the sum of the numbers of the chosen cards. Given this sum, the magician identifies the box from which no card has been chosen. How many ways are there to put all the cards into the boxes so that this trick always works? (Two ways are considered different if at least one card is put into a different box.)

5. Determine if there exists a positive integer n such that n has exactly 2000 prime divisors and $2^n + 1$ is divisible by n.

6. Let $\overline{AH_1}$, $\overline{BH_2}$, and $\overline{CH_3}$ be the altitudes of an acute triangle ABC. The incircle ω of triangle ABC touches the sides BC, CA and AB at T_1, T_2 and T_3, respectively. Consider the symmetric images of the lines H_1H_2, H_2H_3, and H_3H_1 with respect to the lines T_1T_2, T_2T_3, and T_3T_1. Prove that these images form a triangle whose vertices lie on ω.