44^{th} International Mathematical Olympiad

Tokyo, Japan

Day I 9 AM - 1:30 PM

July 13, 2003

1. Let A be a 101-element subset of the set $S = \{1, 2, \ldots, 1000000\}$. Prove that there exist numbers $t_1, t_2, \ldots, t_{100}$ in S such that the sets

$$A_j = \{x + t_j \mid x \in A\} \quad j = 1, 2, \ldots, 100$$

are pairwise disjoint.

2. Determine all pairs of positive integers (a, b) such that

$$\frac{a^2}{2ab^2 - b^3 + 1}$$

is a positive integer.

3. A convex hexagon is given in which any two opposite sides have the following property: the distance between their midpoints is $\sqrt{3}/2$ times the sum of their lengths. Prove that all the angles of the hexagon are equal.

(A convex $ABCDEF$ has three pairs of opposite sides: AB and DE, BC and EF, CD and FA.)

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4. Let $ABCD$ be a cyclic quadrilateral. Let P, Q and R be the feet of perpendiculars from D to lines BC, CA and AB, respectively. Show that $PQ = QR$ if and only if the bisectors of angles ABC and ADC meet on segment AC.

5. Let n be a positive integer and $x_1, x_2, \ldots, x_n$ be real numbers with $x_1 \leq x_2 \leq \ldots \leq x_n$.

   (a) Prove that

   $$\left(\sum_{i=1}^{n}\sum_{j=1}^{n}|x_i - x_j|\right)^2 \leq \frac{2(n^2-1)}{3}\sum_{i=1}^{n}\sum_{j=1}^{n}(x_i - x_j)^2.$$

   (b) Show that the equality holds if and only if $x_1, x_2, \ldots, x_n$ form an arithmetic sequence.

6. Let p be a prime number. Prove that there exists a prime number q such that for every integer n, the number $n^p - p$ is not divisible by q.