22^{nd} United States of America Mathematical Olympiad

April 29, 1993

Time Limit: $3\frac{1}{2}$ hours

1. For each integer $n \geq 2$, determine, with proof, which of the two positive real numbers a and b satisfying
$$a^n = a + 1, \qquad b^{2n} = b + 3a$$
is larger.

2. Let $ABCD$ be a convex quadrilateral such that diagonals AC and BD intersect at right angles, and let E be their intersection. Prove that the reflections of E across AB, BC, CD, DA are concyclic.

3. Consider functions $f : [0, 1] \to \mathbf{R}$ which satisfy
   (i) $f(x) \geq 0$ for all x in $[0, 1]$,
   (ii) $f(1) = 1$,
   (iii) $f(x) + f(y) \leq f(x + y)$ whenever x, y, and $x + y$ are all in $[0, 1]$.

   Find, with proof, the smallest constant c such that
$$f(x) \leq cx$$
for every function f satisfying (i)-(iii) and every x in $[0, 1]$.

4. Let a, b be odd positive integers. Define the sequence (f_n) by putting $f_1 = a$, $f_2 = b$, and by letting f_n for $n \geq 3$ be the greatest odd divisor of $f_{n-1} + f_{n-2}$. Show that f_n is constant for n sufficiently large and determine the eventual value as a function of a and b.

5. Let $a_0, a_1, a_2, \ldots$ be a sequence of positive real numbers satisfying $a_{i-1}a_{i+1} \leq a_i^2$ for $i = 1, 2, 3, \ldots$. (Such a sequence is said to be *log concave*.) Show that for each $n > 1$,
$$\frac{a_0 + \cdots + a_n}{n+1} \cdot \frac{a_1 + \cdots + a_{n-1}}{n-1} \geq \frac{a_0 + \cdots + a_{n-1}}{n} \cdot \frac{a_1 + \cdots + a_n}{n}.$$