23^{rd} United States of America Mathematical Olympiad

April 28, 1994
Time Limit: $3\frac{1}{2}$ hours

1. Let $k_1 < k_2 < k_3 < \cdots$ be positive integers, no two consecutive, and let $s_m = k_1 + k_2 + \cdots + k_m$ for $m = 1, 2, 3, \ldots$. Prove that, for each positive integer n, the interval $[s_n, s_{n+1})$ contains at least one perfect square.

2. The sides of a 99-gon are initially colored so that consecutive sides are red, blue, red, blue, $\ldots$, red, blue, yellow. We make a sequence of modifications in the coloring, changing the color of one side at a time to one of the three given colors (red, blue, yellow), under the constraint that no two adjacent sides may be the same color. By making a sequence of such modifications, is it possible to arrive at the coloring in which consecutive sides are red, blue, red, blue, red, blue, $\ldots$, red, yellow, blue?

3. A convex hexagon $ABCDEF$ is inscribed in a circle such that $AB = CD = EF$ and diagonals AD, BE, and CF are concurrent. Let P be the intersection of AD and CE. Prove that $CP/PE = (AC/CE)^2$.

4. Let $a_1, a_2, a_3, \ldots$ be a sequence of positive real numbers satisfying $\sum_{j=1}^{n} a_j \geq \sqrt{n}$ for all $n \geq 1$. Prove that, for all $n \geq 1$,
$$\sum_{j=1}^{n} a_j^2 > \frac{1}{4}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right).$$

5. Let $|U|$, $\sigma(U)$ and $\pi(U)$ denote the number of elements, the sum, and the product, respectively, of a finite set U of positive integers. (If U is the empty set, $|U| = 0$, $\sigma(U) = 0$, $\pi(U) = 1$.) Let S be a finite set of positive integers. As usual, let $\binom{n}{k}$ denote $\frac{n!}{k!\,(n-k)!}$. Prove that
$$\sum_{U \subseteq S} (-1)^{|U|} \binom{m - \sigma(U)}{|S|} = \pi(S)$$
for all integers $m \geq \sigma(S)$.